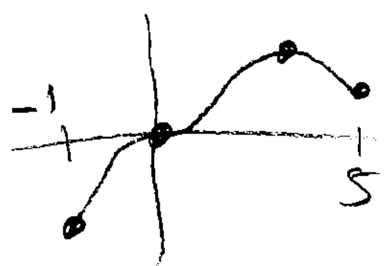
Closing Tues. (Nov. 28th): HW 4.3
Exam 2 is **Tuesday**!!!
covers 3.1-3.6, 3.9-3.10, 10.2, 4.1
Expect: 6 pages
Page 1: Find Deriv./Slope/Tangent
Page 2-4: Implicit, Parametric, Linear Approx., Abs. Max/Min
Page 5-6: Related Rates (Expect to see at least one picture/question directly HW).

Office Hours today: 1:30-3:00 COM B-006

Entry Task:
Find the abs. max and min of $f(x) = x^3 e^{-x}$ on $[-1,5]$.



$$f'(x) = -x^3 e^{-x} + 3x^2 e^{-x}$$
$$= x^2 e^{-x}(-x+3) \stackrel{?}{=} 0$$

$x^2 = 0$ OR $e^{-x} = 0$ OR $-x+3=0$
$x = 0$ NEVER $x = 3$

ENDPOINTS:
$x = -1 \Rightarrow f(-1) = (-1)^3 e^1 = -e \approx -2.7182$
$x = 5 \Rightarrow f(5) = 5^3 e^{-5} = \frac{125}{e^5} \approx 0.8422$

CRITICAL NUMBERS
$x = 0 \to f(0) = 0^3 e^{-0} = 0 \cdot 1 = 0$
$x = 3 \to f(3) = 3^3 e^{-3} = \frac{27}{e^3} \approx 1.3443$

$$\boxed{\begin{array}{l} \text{ABS. MIN} = -e \\ \text{ABS. MAX} = \dfrac{27}{e^3} \end{array}}$$

4.3 Classifying Critical Points (Local Max/Min)

Recall:

$y = f(x)$	$y' = f'(x)$
horiz. tangent	zero
increasing	positive
decreasing	negative
vertical tangent, sharp corner, or not continuous	does not exist

Key, big, essential observation

(*First derivative test*)

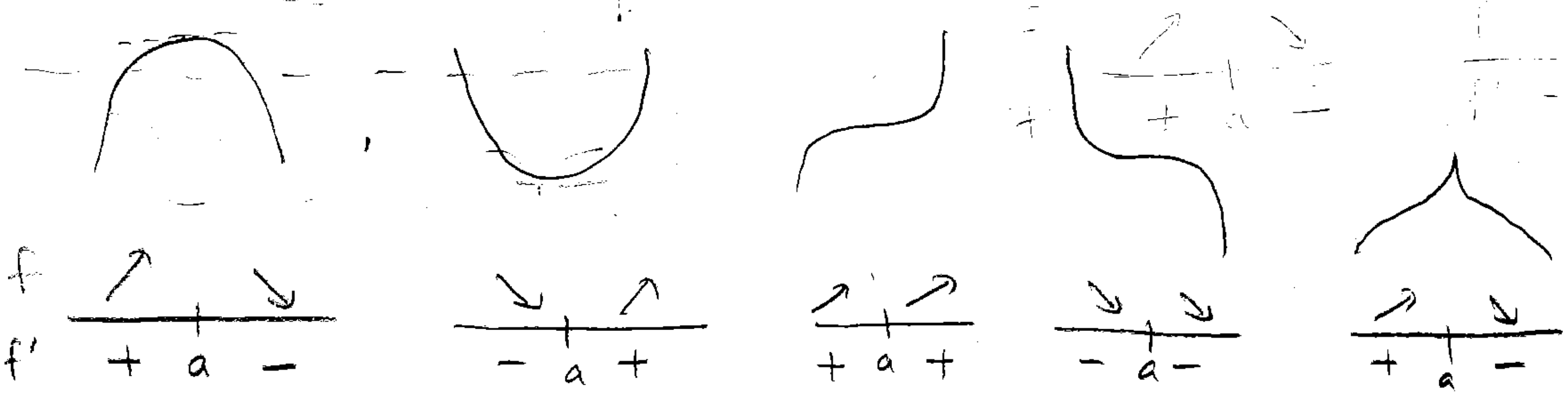
If $x = a$ is a critical number for $f(x)$

AND

if $f'(x)$ changes from…

1. …positive to negative, then a **local maximum** occurs at $x = a$.

2. …negative to positive, then a **local minimum** occurs at $x = a$.



Example: Find and classify the critical numbers for

$$y = x^3 + 3x^2 - 72x$$

$$y' = 3x^2 + 6x - 72 \stackrel{?}{=} 0$$

$$3(x^2 + 2x - 24) = 0$$

$$3(x+6)(x-4) = 0$$

$$x = -6 \quad \text{or} \quad x = 4$$

$$\boxed{f'(x) = 3(x+6)(x-4)}$$

f	$\nearrow$		$\searrow$		$\nearrow$
f'	$+$	-6	$-$	4	$+$

For $x < -6$: $3(x+6)(x-4)$, $-$ $-$

For $-6 < x < 4$: $3(x+6)(x-4)$, $+$ $-$

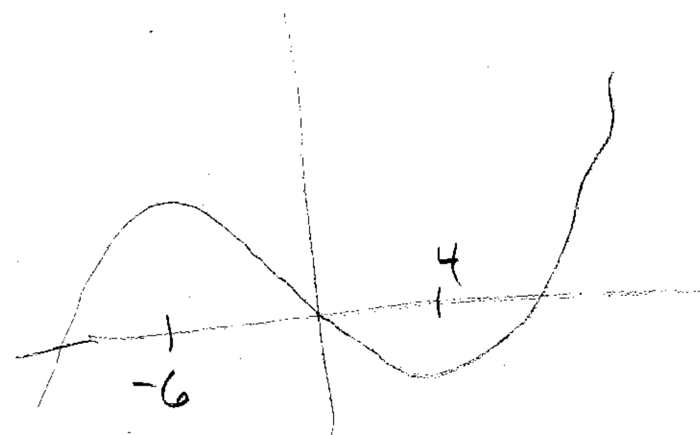
For $x > 4$: $3(x+6)(x-4)$, $+$ $+$

THUS

$x = -6$ GIVES A LOCAL <u>MAX.</u>

AND

$x = 4$ GIVES A LOCAL <u>MIN</u>.

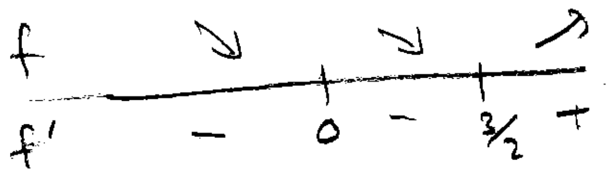


Example: Find and classify the critical numbers for

$$y = x^4 - 2x^3$$

$y' = 4x^3 - 6x^2 = 2x^2(2x-3) \stackrel{?}{=} 0$

$x = 0$ or $x = 3/2$



$y' = 2x^2(2x-3)$

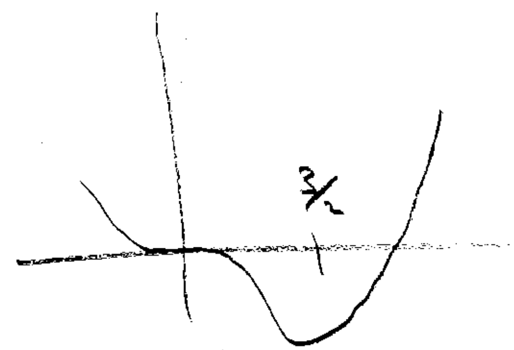
$x < 0 \Rightarrow (+)(-) = -$

$0 < x < 3/2 \Rightarrow (+)(-) = -$

$\frac{3}{2} < x \Rightarrow (+)(+) = +$

$x = 0$ IS <u>NOT</u> A LOCAL MAX OR MIN

$x = 3/2$ IS A LOCAL <u>MIN</u>



Example: Find and classify the critical numbers for

$$y = x^{2/3}$$

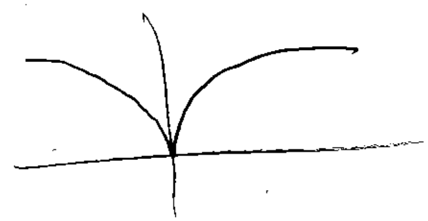
$$y' = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}}$$

$\frac{2}{3x^{1/3}} = 0$ HAS <u>NO</u> SOL'N

BUT $\frac{2}{3x^{1/3}}$ DOES NOT EXIST AT $x = 0$ WHICH IS IN THE DOMAIN

$x = 0$ GIVES A LOCAL MIN

y	↘		↗
y'	$-$	0	$+$



$x < 0 \Rightarrow y' = \frac{2}{3x^{1/3}}$ IS NEGATIVE

$x > 0 \Rightarrow y' = \frac{2}{3x^{1/3}}$ IS POSITIVE

Example: Find and classify the critical numbers for

$$y = \frac{x^3}{x^2 - 1}$$

DOMAIN
$x \neq -1$
$x \neq 1$

$$y' = \frac{(x^2-1)3x^2 - x^3 \cdot (2x)}{(x^2-1)^2}$$

$$y' = \frac{3(x^2-1)x^2 - 2x^4}{(x^2-1)^2}$$

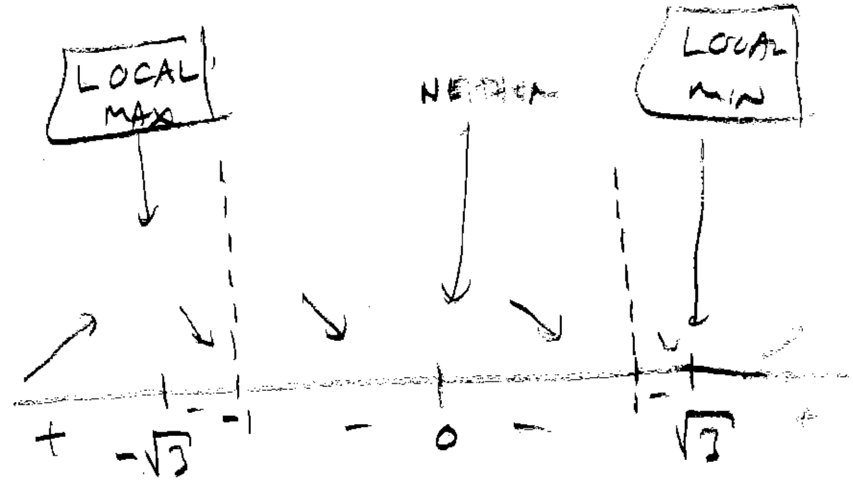
$$= \frac{x^2[3x^2 - 3 - 2x^2]}{(x^2-1)^2}$$

$$y' = \frac{x^2[x^2-3]}{(x^2-1)^2} \overset{?}{=} 0$$

$$x^2(x^2-3) \overset{?}{=} 0$$

$x = 0$ or $x^2 - 3 = 0$

$x = \pm\sqrt{3}$



$$y' = \frac{x^2(x^2-3)}{(x^2-1)^2}$$

$x < -\sqrt{3} \Rightarrow +$

$-\sqrt{3} < x < 0 \Rightarrow -$

$0 < x < \sqrt{3} \Rightarrow -$

$x > \sqrt{3} \Rightarrow +$

The 2nd Derivative

$$y'' = f''(x) = \frac{d}{dx}\left(f'(x)\right)$$

= "rate of change of first derivative"

Terminology

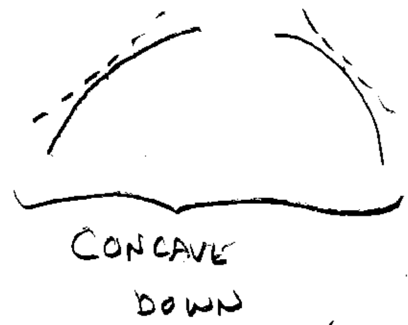
If **f''(x) is positive**,
then the **slope of f(x) is increasing**
and we say f(x) is **concave up**.

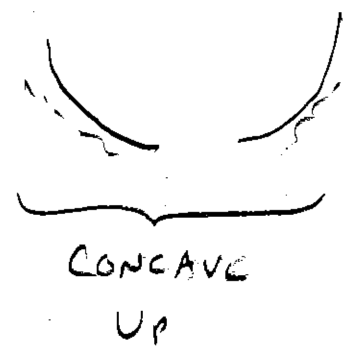
If **f''(x) is negative**,
then the **slope of f(x) is decreasing**
and we say f(x) is **concave down**.

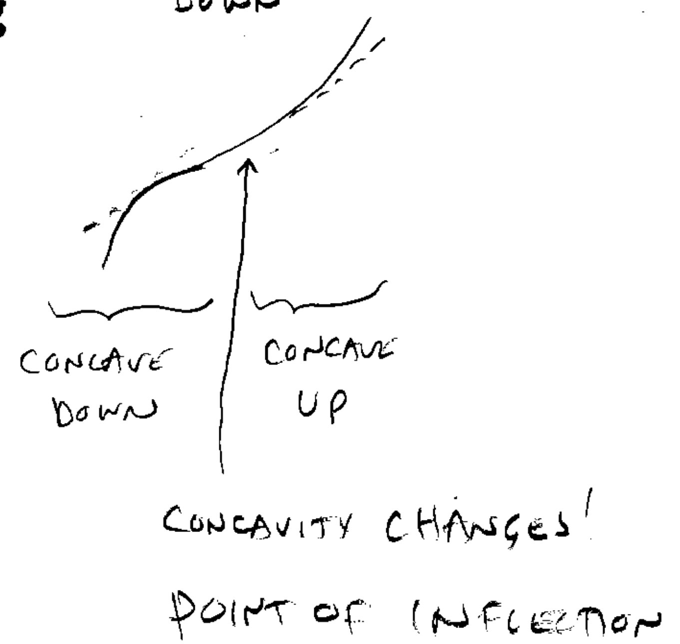
A point in the domain of the function at which the concavity changes is called an **inflection point**.

Summary:

$y = f(x)$	$y'' = f''(x)$
possible inflection	zero
concave up	Positive
concave down	Negative
possible inflection	does not exist







Example: Find all inflection points and indicate where function is concave up and concave down for

$$y = x^4 - 2x^3$$

$$y' = 4x^3 - 6x^2$$

$$y'' = 12x^2 - 12x \stackrel{?}{=} 0$$

$$12x(x-1) \stackrel{?}{=} 0$$

$x = 0$ or $x = 1$

y	$\cup$		$\cap$		$\cup$
y''	$+$	0	$-$	1	$+$

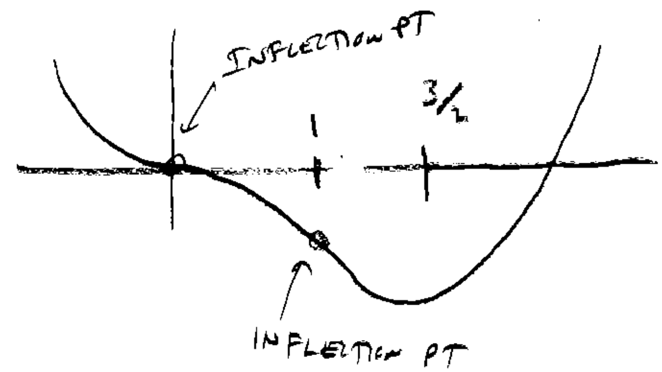
$x < 0 \Rightarrow 12x(x-1) \quad (-)(-) \rightarrow +$

$0 < x < 1 \Rightarrow 12x(x-1) \quad (+)(-) \rightarrow -$

$x > 1 \Rightarrow 12x(x-1) \quad (+)(+) \rightarrow +$

SAME EXAMPLE
FROM EARLIER

y	$\searrow$		$\searrow$		$\nearrow$
y'	$-$	0	$-$	$3/2$	$+$
	$\cup$		$\cap$		$\cup$
	$+$	0	$-$	1	$+$



Clever Observation

(*Second Derivative Test*)

If $x = a$ is a critical number for $f(x)$

AND

1. if $f''(a)$ is positive (CCU),
   then a local min occurs at $x = a$.

2. if $f''(a)$ is negative (CCU),
   then a local max occurs at $x = a$.

3. if $f''(a) = 0$,
   then we say the 2nd deriv. test is
   inconclusive (need other method)

Example: Find and classify the critical numbers for

$$y = 2 + 2x^2 - x^4$$

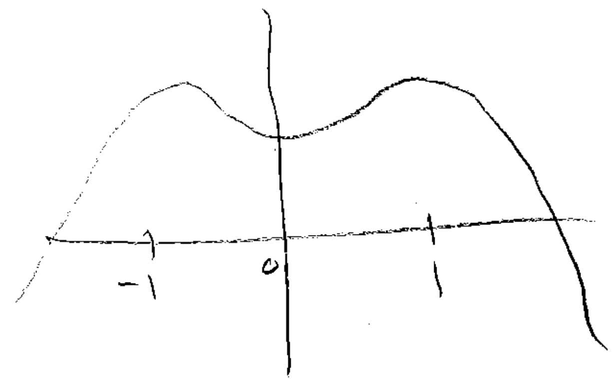
(use the 2nd deriv. test)

$$y' = 4x - 4x^3 \stackrel{?}{=} 0$$

$$4x(1 - x^2) \stackrel{?}{=} 0$$

$$4x(1-x)(1+x) \stackrel{?}{=} 0$$

$x = 0$, $x = 1$, or $x = -1$



$$y'' = 4 - 12x^2$$

$x = 0 \rightarrow y''(0) = 4$, CONCAVE UP $\boxed{\text{LOCAL MIN}}$

$x = -1 \rightarrow y''(-1) = 4 - 12 = -8$, CONCAVE DOWN $\boxed{\text{LOCAL MAX}}$

$x = 1 \rightarrow y''(1) = 4 - 12 = -8$, CONCAVE DOWN $\boxed{\text{LOCAL MAX}}$